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A JUSTIFICATION OF THE KdV APPROXIMATION
TO FIRST ORDER IN THE CASE OF N-SOLITON
WATER WAVES IN A CANAL

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April 1981

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Technical Summary Report # 2208

April 1981

ABSTRACT

·We consider the Euler equations for a perfect fluid in a flat-
bottomed canal in the time-dependent case. A formal expansion procedure
for small amplitude, long waves analogous to that of Friedrichs and Hyers
for solitary waves is developed and leads to the Korteweg-de Vries equa-
tion (KdV for short) for the lowest order term. The higher order terms in
the expansion satisfy the inhomogeneous version of the linearized KdV
equation.

Of particular interest to us are those solutions of the KdV equation
called N-solitons, which asymptotically separate into N travelling waves
with distinct speeds. Using certain facts about the linearized KdV
equation and some properties of the N-solitons, we prove that the next
term in this expansion can be uniquely specified by certain asymptotic
conditions and a symmetry requirement. This solution behaves like an
N-soliton; asymptotically, it separates into N travelling waves with the
same speeds and phases as those of the leading term.

AMS (MOS) Subject Classification: 76B15; 35Q20; 35C05; 35G30

Key Words: Euler equations; Korteweg-de Vries equation; N-solitons

Work Unit Numbers 1 (Applied Analysis)
2 (Physical Mathematics)
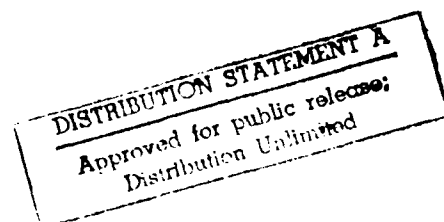
SIGNIFICANCE AND EXPLANATION

The Euler equations for fluid flow in a flat-bottomed canal may be approximated, in the long wavelength, small amplitude limit, by the Korteweg-de Vries equation (KdV for short). We study the nature of this approximation and attempt to justify it mathematically. The Friedrichs and Hyers proof [9] of the existence of the solitary wave provides such a justification for travelling wave solutions of the Euler equations. In fact, they gave a formal expansion procedure for the solution which yields the (time-independent) KdV equation for the leading term.

In this paper, time dependent solutions of the Euler equations are examined in the same long wavelength limit. A formal expansion procedure analogous to that of Friedrichs and Hyers is developed, and the full KdV equation is derived for the leading term. Choosing the so-called N-soliton solution of the KdV equation, we seek solutions to the Euler equations with similar properties. The N-solitons are non-linear superpositions of N solitary waves with distinct speeds, and thus, for large positive or negative times, they decouple into N travelling waves. By analyzing the linearized KdV equation, we show that with suitable boundary conditions and a symmetry condition, the next term in the formal expansion is unique and resembles an N soliton. The effect of the first-order correction to the KdV equation is merely to alter the shape of the waves slightly; their speeds and phases remain the same.

A JUSTIFICATION OF THE KdV APPROXIMATION
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1. INTRODUCTION

The Korteweg-de Vries equation (KdV for short) was originally derived
in 1895 as an approximation for fluid flow in a flat-bottomed canal [14].
This non-linear evolution equation for a function of one space variable
has the rather remarkable property, discovered by Gardner, Greene, Kruskal,
and Miura [10], that it may be solved more or less exactly. In fact, a
Hamiltonian structure can be introduced and the KdV equation may be
regarded as a completely integrable Hamiltonian system. One very
interesting class of solutions is the set of so-called N-solitons.
These solutions behave, for large positive and negative times, like N
exponentially decreasing 'bumps' moving at distinct speeds. A natural
question to ask is whether such "N-tuple waves" exist for the full set of
Euler equations governing the fluid flow in a canal.

For N = 1, such wave solutions, known as solitary waves, do in
fact exist [3,4,9]. In [9], Friedrichs and Hyers gave a formal expansion
procedure for the Euler equations in which a time-independent form of the
KdV equation arose as the equation satisfied by the lowest order term.
The higher order terms of their expansion satisfied the inhomogeneous
form of the linearization of the non-linear ordinary differential equation
for the leading term. With a symmetry condition added to the requirement
of exponential decay, this equation could be solved uniquely. After re-
formulating the problem, the convergence of this formal solution was
shown by the implicit function theorem. Later Beale [4] simplified the
argument by using a generalized implicit function theorem due to Zehnder

[25]. In both of these approaches, the time-independent nature of the problem is relied upon from the beginning.

In attempting to generalize these results to N-solitons for $N \geq 2$, the problem becomes unavoidably time-dependent. An essentially trivial step in both approaches to the solitary wave problem, namely inverting the linearized KdV operator, now becomes a serious difficulty. Constructing a formal solution which behaves like a N-soliton requires solving the inhomogeneous linearized KdV equation with prescribed asymptotic behavior. We do this for the first order correction term by using the explicit form of the inhomogeneous term. For higher order corrections, the existence of some solution is guaranteed by Duhamel's principle and the solvability of the Cauchy problem for the linearized KdV equation [19]. However, in such an approach, initial values (say at $t = 0$) 'parameterize' the set of all solutions and we cannot as yet single out those solutions with the desired asymptotic behavior.

In this paper we present the following results:

(i) The time-dependent analogue of the formal expansion of Friedrichs-Hyers [9] is developed. For perturbations of a steady horizontal flow with Froude number near 1 which are of small amplitude, long wavelength, and slow time variation, we consider a formal power series solution of the Euler equations. The small parameter ε is related to the Froude number. As in [9], the leading term satisfies the KdV equation and the higher order terms satisfy the inhomogeneous linearized KdV equation. However, in this case, both of these equations are time-dependent.

(ii) Using results on the solvability of the Cauchy problem for the linearized KdV equation [19] and certain facts about N-solitons, we analyze the first order term completely. In particular, we

show that this term is uniquely determined by the following
conditions:

(a) (symmetry) $u(x,t) = u(-x,-t)$

(b) (asymptotic decay in moving frames)

$u(ct + \xi,t) \to 0$ exponentially fast as

$t \to +\infty$ for ξ fixed unless $c = c_j$

$j = 1,\ldots N$ where $\{c_j\}$ are the N

soliton speeds

(c) (asymptotic shape)

$\lim_{t \to \infty} u(c_j t + \xi ,t)$ is an exponentially

decreasing function of ξ.

This is the sense in which we use the term justification in the title of
this paper. The first order correction to the KdV N-soliton, as chosen
above, does not alter any of the essential features of the solution.
After a long time, the water wave decomposes into N travelling waves
with distinct speeds, each of which is exponentially decreasing in
space when viewed from the appropriate moving frame of reference.

Section 2 contains the time-dependent analogue of the formal
expansion of Friedrichs and Hyers [9] as well as the mapping formulation
of Beale [4]. For the latter set-up, invertibility of the linearized
mapping at $\varepsilon = 0$ is shown in the formal sense provided the linearized
KdV operator is invertible. The basic facts concerning the Cauchy
problem for the linearized KdV equation are presented in Section 3.
Explicit solvability for this problem is related to the so-called
inverse scattering method for solving the KdV equation [10, 19]. Using
certain facts about N-solitons, which we present in the Appendix, and
the particular terms arising in the expansion of Section 2, the first-
order correction to the N-soliton is analyzed in Section 4.

2. THE EULER EQUATIONS, THE KdV LIMIT, AND A FORMAL EXPANSION

In dimensionless variables, the Euler equations for a perfect fluid in a two-dimensional, flat-bottomed domain D, with a free boundary $y = \Gamma(t,x)$ as upper surface, subject only to gravitational acceleration g are (cf. Stoker [21]):

$$(2.1) \begin{cases} \text{(i)} \quad \phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad D \equiv \{(x,y) : 0 < y < \Gamma(t,x)\} \\[2ex] \text{(ii)} \quad \phi_y = 0 \qquad \text{along} \quad y = 0 \\[2ex] \text{(iii)} \quad \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + \gamma y = \text{constant} \\[1ex] \qquad\qquad\qquad\qquad \text{along} \quad y = \Gamma(t,x) \\[2ex] \text{(iv)} \quad \Gamma_t + \phi_x \cdot \Gamma_x - \phi_y = 0 \quad \text{along} \quad y = \Gamma(t,x) \end{cases}$$

where $\phi = \phi(x,y,t)$ is the velocity potential and $\gamma \equiv \dfrac{gh}{U^2}$ where h is the length scaling and U is the velocity scaling.

$\gamma^{-1/2}$ is called the Froude number or reduced depth and is a parameter of the problem. The linear theory of water waves [21] predicts $\gamma = 1$, while the existence of solitary waves occurs for $\gamma < 1$ but sufficiently close to 1. From now on, we assume

(2.2) $0 < 1 - \gamma \ll \gamma < 1$ and in fact, we define a small parameter ε by the equation: $\gamma = e^{-3\varepsilon}$.

In this section, we will consider flows which are very nearly the trivial flow of constant horizontal speed 1 given by the solution $\phi = x$, $\Gamma = 1$, $\gamma = 1$ of (2.1). Introducing auxiliary variables ξ', η' which vary over a fixed horizontal strip $0 < \eta' < 1$, we may eliminate the unknown free surface at the expense of defining x, y as functions of ξ', η', t. In steady flow problems, t does not appear and $\xi' + i\eta'$ is usually the complex potential function, but for time-dependent

problems, we express both the potential function ϕ and the physical

coordinates x,y in terms of ξ', η' and t. Provided the mapping

$(\xi',\eta') \mapsto (x,y)$ is invertible for every t, solving the problem in the

ξ', η' plane is equivalent to solving the original system in the x,y

plane. In the neighborhood of the trivial horizontal flow, this mapping

is roughly the identity map, hence it will be invertible.

After expressing the problem in these new independent variables, a

new dependent complex variable, $\lambda' - i\theta'$, defined as the logarithm of

the complex velocity W $(W \equiv \phi_x - i\phi_y)$, will be introduced. By different-

iating with respect to ξ' along $\eta' = 1$, ϕ is eliminated and a new

system of equations for x, y, λ', θ' is obtained. Defining a small

parameter $a \equiv \varepsilon^{1/2}$, we rescale the independent variables ξ', η', t and

the small dependent variables λ', θ', $x' \equiv x-\xi'$, $y' \equiv y-\eta'$. The system (2.1)

in the rescaled variables, ξ, η, τ and $\hat{x}$, $\hat{y}$, λ, θ respectively, becomes:

$$
\begin{cases}
\text{(i)} \quad (\varepsilon^{1/2}\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta})(\hat{x} + i\varepsilon^{1/2}\hat{y}) = (\varepsilon^{1/2}\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta})(\lambda - i\varepsilon^{1/2}\theta) \\[2mm]
\qquad\qquad\qquad\qquad = 0 \quad \text{in} \quad 0 < \eta < 1 \\[2mm]
\qquad\qquad\qquad \text{(scaled Cauchy-Riemann equation)} \\[3mm]
\text{(ii)} \quad \theta = 0 ; \quad \hat{y} = 0 \quad \text{along} \quad \eta = 0 \\[3mm]
\text{(iii)} \quad e^{\varepsilon\lambda}\{\cos(\varepsilon^{3/2}\theta)[\varepsilon\lambda_\tau + \varepsilon^2(\lambda_\tau\hat{x}_\xi - \lambda_\xi\hat{x}_\tau) + \varepsilon^3(\theta_\xi\hat{y}_\tau - \theta_\xi\hat{y}_\tau)] \\[2mm]
\qquad\qquad + \frac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}}[-\varepsilon^3\theta_\tau + \varepsilon^4(\lambda_\tau\hat{y}_\xi - \lambda_\xi\hat{y}_\tau + \theta_\xi\hat{x}_\tau - \theta_\tau\hat{x}_\xi)]\} \\[2mm]
\qquad\qquad + e^{2\varepsilon\lambda}\lambda_\xi + e^{-3\varepsilon}\hat{y} = 0 \quad \text{along} \quad \eta = 1 \\[3mm]
\qquad\qquad\qquad \text{(Bernoulli's law)} \\[3mm]
\text{(iv)} \quad e^{\varepsilon\lambda}\hat{y}_\xi\cos(\varepsilon^{3/2}\theta) - e^{\varepsilon\lambda}(1+\varepsilon\hat{x}_\xi)\cdot\frac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}} \\[2mm]
\qquad\qquad + \varepsilon\hat{y}_\tau + \varepsilon^2(\hat{x}_\xi\hat{y}_\tau - \hat{x}_\tau\hat{y}_\xi) = 0 \quad \text{along} \quad \eta = 1 \\[2mm]
\qquad\qquad\qquad \text{(free boundary/streamline condition)}
\end{cases}
$$

(2.3)

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We proceed to derive system (2.3) below and then discuss a formal

expansion procedure using power series in ε. A mapping formalism, as in

Beale [4], is also presented and formal invertibility of the linearized

map at $\varepsilon = 0$ is examined.

A.   Reformulation via a Conformal Mapping

Introduce complex variables into (2.1) as follows:

$z \equiv x+iy$, $F(z,t) \equiv \phi(z,t) + i\psi(z,t)$. The complex velocity

$W(z,t) \equiv F_z(z,t) = \phi_x - i\phi_y$ so $\operatorname{Re} W = \phi_x$, the horizontal velocity, and

$-\operatorname{Im} W = \phi_y$, the vertical velocity. Since we are considering flows near

the trivial one, for which the free surface is $\Gamma(t,x) \equiv 1$, we assume that

there exists a complex variable $\zeta' = \xi' + i\eta'$ defined on the fixed strip

$\{(\xi',\eta') \mid 0 < \eta' < 1\}$ and a conformal mapping $z = z(\zeta',t)$ such that

the boundaries of the flow domain, $y = 0$ and $y = \Gamma(t,x)$ (where $y = \operatorname{Im} z$)

correspond to the boundaries $\eta' = 0$, $\eta' = 1$ respectively.

Given the existence of such a mapping, we define new dependent

variables implicitly:

$$(2.4) \quad \begin{cases} f(\zeta',t) \equiv F(z(\zeta',t),t) \quad ; \\[1mm] w(\zeta',t) \equiv W(z(\zeta',t),t) \quad \text{so that} \\[1mm] w(\zeta',t) = \dfrac{f_{\zeta'}}{z_{\zeta'}} \end{cases}$$

Thus ζ' derivatives of f are expressible in terms of w and z.

Substitution in (2.1) and differentiation with respect to ξ' along $\eta' = 1$

yields a system with w,z as dependent variables, namely:

(2.5)
$\quad$ (i) $\quad w(\zeta',t)$ and $z(\zeta',t)$ are holomorphic functions of

$\quad\quad\quad$ ζ' in $0 < \text{Im } \zeta' < 1$

$\quad$ (ii) $\quad \text{Im } w = 0$; $\text{Im } z = 0$ along $\eta' = 0$

$\quad$ (iii) $\quad \text{Re } (w_t z_{\xi'} - w_{\xi'} z_t) + \frac{1}{2} (|w|^2)_{\xi'} + \gamma \text{ Im } (z_{\xi'}) = 0$

$\quad\quad\quad$ along $\eta' = 1$

$\quad$ (iv) $\quad \text{Im } (z_t / z_{\xi'}) + \text{Im } (w/\bar{z}_{\xi'}) = 0$ along $\eta' = 1$

$\quad\quad\quad$ (This last condition comes from the relations $\Gamma_x = y_{\xi'}/x_{\xi'}$;

$\quad\quad\quad$ $\Gamma_t = y_t - x_t y_{\xi'}/x_{\xi'}$ on $\eta' = 1$.)

It is convenient to replace w by $\lambda' - i\theta'$, defined by the relation

(2.6)
$$w = e^{\lambda' - i\theta'} .$$

This substitution was introduced by Levi-Civita [16] in the periodic case
of infinite depth; it has the virtues of simplifying the $|w|$ differentiation,
ensuring $w \neq 0$ for any solution, and making $\lambda' - i\theta' \equiv 0$ the trivial
flow. Upon substitution, we obtain:

(2.7)
$\quad$ (i) $\quad z(\zeta',t)$, $(\lambda'-i\theta')(\zeta',t)$ are holomorphic in

$\quad\quad\quad$ ζ' for $0 < \text{Im } \zeta' < 1$

$\quad$ (ii) $\quad \theta' = 0$, $y = 0$ along $\eta' = 0$

$\quad$ (iii) $\quad \text{Re } (e^{\lambda'-i\theta'}[\lambda'_t - i\theta'_t)z_{\xi'} - (\lambda'_{\xi'} - i\theta'_{\xi'})z_t])$

$\quad\quad\quad$ $+ e^{2\lambda'}\lambda'_{\xi'} + \gamma \text{ Im } z_{\xi'} = 0$ along $\eta' = 1$

$\quad$ (iv) $\quad \text{Im } (z_t/z_{\xi'}) + \text{Im } (e^{\lambda'-i\theta'}/\bar{z}_{\xi'}) = 0$ along $\eta' = 1$

We note that this is a system of equations for two holomorphic functions on a strip which are real for ζ' real (the bottom) and satisfy a pair of coupled nonlinear time-dependent boundary conditions along the top of the strip. Kano and Nishida [11] used essentially the system (2.5), along with some basic facts about harmonically conjugate functions on a strip, to obtain a nonlinear expression for the t-derivatives of x and ϕ along $\eta' = 1$, for which a solution will exist to the Cauchy problem for small times (see also [18]).

We will now consider a particular limiting case of system (2.7) corresponding to long wavelength, small amplitude waves of slow time variation and will obtain the Korteweg-deVries equation in the limit. We assume that, as $|\zeta'| \to \infty$, $\lambda' - i\theta' \to 0$ and $z_{\zeta'} \to 1$, so the perturbations from the steady flow vanish asymptotically. The limiting case is given by the following rescaling:

Define new independent variables

(2.8) $\xi = a\xi'$; $\eta = \eta'$; $\tau = a^3 t$, where $a^2 = \varepsilon$,

and new dependent variables $\hat{x}, \hat{y}, \theta, \lambda$ by:

(2.9) $\begin{cases} a\hat{x}(\xi,\eta,\tau,\varepsilon) & = & \mathrm{Re} \ (z(\zeta',t)) - \xi' \\[2mm] a^2\hat{y}(\xi,\eta,\tau,\varepsilon) & = & \mathrm{Im} \ (z(\zeta',t) \ - \eta' \\[2mm] a^2\lambda(\xi,\eta,\tau,\varepsilon) & = & \lambda'(\zeta',t) \\[2mm] a^3\theta(\xi,\eta,\tau,\varepsilon) & = & \theta'(\zeta',t) \end{cases}$

Substituting these variables into the system (2.7) gives system (2.3) above, which we have therefore derived.

B. A Formal Solution Procedure

If we consider the system (2.3) and assume expansions for $\lambda, \theta, \hat{x}, \hat{y}$ of the form:

$$(2.10) \quad \begin{cases} \lambda(\xi,\eta,\tau,\varepsilon) &= \displaystyle\sum_{j=0}^{\infty} \lambda^{(j)}(\xi,\eta,\tau) \, \varepsilon^j \\[2em] \theta(\xi,\eta,\tau,\varepsilon) &= \displaystyle\sum_{j=0}^{\infty} \theta^{(j)}(\xi,\eta,\tau) \, \varepsilon^j \\[2em] \hat{x}(\xi,\eta,\tau,\varepsilon) &= \displaystyle\sum_{j=0}^{\infty} \hat{x}^{(j)}(\xi,\eta,\tau) \, \varepsilon^j \\[2em] \hat{y}(\xi,\eta,\tau,\varepsilon) &= \displaystyle\sum_{j=0}^{\infty} \hat{y}^{(j)}(\xi,\eta,\tau) \, \varepsilon^j \end{cases}$$

then (i) of (2.3) implies the system:

$$(2.11) \quad \begin{cases} \lambda_\xi^{(j)} + \theta_\eta^{(j)} = 0 \; ; \quad -\lambda_\eta^{(j)} + \theta_\xi^{(j-1)} = 0 \\[1em] \hat{x}_\xi^{(j)} + \hat{y}_\eta^{(j)} = 0 \; ; \quad -\hat{x}_\eta^{(j)} + \hat{y}_\xi^{(j-1)} = 0 \end{cases}$$

In particular, $\lambda_\eta^{(0)} = 0$; $\hat{x}_\eta^{(0)} = 0$. From the boundary condition (ii) on $\theta, \hat{y}$ at $\eta = 0$, this implies:

$$(2.12) \quad \begin{cases} \lambda^{(0)} = \lambda^{(0)}(\xi,\tau) \; ; \qquad \hat{x}^{(0)} = \hat{x}^{(0)}(\xi,\tau) \; ; \\[1em] \theta^{(0)} = -\lambda_\xi^{(0)}(\xi,\tau) \, \eta \; ; \quad \hat{y}^{(0)} = \hat{x}_\xi^{(0)}(\xi,\tau) \, \eta \end{cases}$$

so
$$\lambda^{(1)}(\xi,\eta,\tau) = -\frac{1}{2}\lambda^{(0)}_{\xi\xi}(\xi,\tau)\,\eta^2 + Q^{(1)}(\xi,\tau)$$

$$\hat{x}^{(1)}(\xi,\eta,\tau) = -\frac{1}{2}\hat{x}^{(0)}_{\xi\xi}(\xi,\tau)\,\eta^2 + P^{(1)}(\xi,\tau)$$

$$\theta^{(1)}(\xi,\eta,\tau) = \frac{1}{6}\lambda^{(0)}_{\xi\xi\xi}(\xi,\tau)\,\eta^3 - Q^{(1)}_{\xi}(\xi,\tau)\,\eta$$

$$\hat{y}^{(1)}(\xi,\eta,\tau) = -\frac{1}{6}\hat{x}^{(0)}_{\xi\xi\xi}(\xi,\tau)\,\eta^3 + P^{(1)}_{\xi}(\xi,\tau)\,\eta$$

Proceeding inductively, with $Q^{(0)} \equiv \lambda^{(0)}$, $P^{(0)} \equiv \hat{x}^{(0)}$, we have

$$(2.13)\quad
\begin{cases}
\lambda^{(k)}(\xi,\eta,\tau) = \displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j\eta^{2j}}{(2j)!}\left(\frac{\partial}{\partial\xi}\right)^{2j}Q^{(k-j)}(\xi,\tau) \\[3mm]
\hat{x}^{(k)}(\xi,\eta,\tau) + \displaystyle\sum_{j=o}^{\infty}\frac{(-1)^j\eta^{2j}}{(2j)!}\left(\frac{\partial}{\partial\xi}\right)^{2j}P^{(k-j)}(\xi,\tau)
\end{cases}$$

with similar expressions for $\theta^{(j)}$, $\hat{y}^{(j)}$ involving odd powers of η and $\partial/\partial\xi$.

Thus if $\lambda^{(j)}$, $j < k$, are known, and similarly for $\hat{x}^{(j)}$, there are two unknown functions $P^{(k)}(\xi,\tau)$, $Q^{(k)}(\xi,\tau)$ which arise in terms of order ε^k and higher. Substituting these series (2.13) into the two boundary conditions at $\eta = 1$, namely

$$\lambda_\xi\,e^{2\lambda} + e^{-3\lambda}\,\hat{y}_\xi + e^{\varepsilon\lambda}\Big\{\cos(\varepsilon^{3/2}\theta)\,[\varepsilon\lambda_\tau + \varepsilon^2(\lambda_\tau\hat{x}_\xi - \lambda_\xi\hat{x}_\tau) + \varepsilon^3(\theta_\tau\hat{y}_\xi - \theta_\xi\hat{y}_\tau)]$$

$$+ \frac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}}\,[-\varepsilon^3\theta_\tau + \varepsilon^4(\lambda_\tau\hat{y}_\xi - \lambda_\xi\hat{y}_\tau - \theta_\tau\hat{x}_\xi + \theta_\xi\hat{x}_\tau)]\Big\} = 0$$

$$\text{along}\quad \eta = 1$$

and

$$\hat{y}_\xi\cos(\varepsilon^{3/2}\theta) - (1+\varepsilon\hat{x}_\xi)\frac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}} + e^{-\varepsilon\lambda}\Big\{\varepsilon\hat{y}_\tau + \varepsilon^2(\hat{x}_\xi\hat{y}_\tau - \hat{x}_\tau\hat{y}_\xi)\Big\} = 0 \quad \text{along}\quad \eta = 1$$

we obtain, setting $\varepsilon = 0$,

$$(2.14)\qquad\qquad Q^{(0)}_\xi + P^{(0)}_{\xi\xi} = 0$$

from each equation.

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Terms of order ϵ in the boundary conditions at $\eta = 1$ are:

$$(2.15) \quad \begin{cases} \lambda_\xi^{(1)} + 2\lambda^{(0)}\lambda_\xi^{(0)} + \hat{y}_\xi^{(1)} - 3\hat{y}_\xi^{(0)} + \lambda_\tau^{(0)} = 0 \\ \hat{y}_\xi^{(1)} - \theta^{(1)} - \hat{x}_\xi^{(0)}\theta^{(0)} + \hat{y}_\tau^{(0)} = 0 \end{cases}$$

which imply:

$$-\frac{1}{2}Q_{\xi\xi\xi}^{(0)} + Q_\xi^{(1)} + 2Q^{(0)}Q_\xi^{(0)} - \frac{1}{6}P_{\xi\xi\xi}^{(0)} + P_{\xi\xi}^{(1)} - 3P_{\xi\xi}^{(0)} + Q_\tau^{(0)} = 0$$

$$P_{\xi\xi}^{(1)} - \frac{1}{6}P_{\xi\xi\xi\xi}^{(0)} - \frac{1}{6}Q_{\xi\xi\xi}^{(0)} + Q_\xi^{(1)} + P_\xi^{(0)}Q_\xi^{(0)} + P_{\xi\tau}^{(0)} = 0$$

so that $P_{\xi\xi}^{(1)} + Q_\xi^{(1)}$ is known in terms of $Q^{(0)}$, $P^{(0)}$ and drops out upon subtracting these two equations. If we integrate (2.14), we have $Q^{(0)} + P_\xi^{(0)} = 0$ (by our boundary conditions as $|\xi| \to \infty$), so that the two boundary conditions for order ϵ imply:

$$(2.16) \quad 2Q_\tau^{(0)} + 3Q^{(0)}Q_\xi^{(0)} + 3Q_\xi^{(0)} - \frac{1}{3}Q_{\xi\xi\xi}^{(0)} = 0$$

which is a form of the KdV equation.

Remark: The formal expansion of Friedrichs and Hyers [9] for the solitary wave has the time-independent form of (2.16) as the equation for the leading term.

If we pick any $Q^{(0)}$ satisfying (2.16), we obtain $P^{(0)}$ by integration, since $P_\xi^{(0)} = -Q^{(0)}$.

The order ϵ^k terms in the boundary conditions yield two equations of the form:

$$(2.17) \quad \begin{cases} \lambda_\xi^{(k)} + 2\lambda^{(0)}\lambda_\xi^{(k-1)} + 2\lambda^{(k-1)}\lambda_\xi^{(0)} + \hat{y}_\xi^{(k)} - 3\hat{y}_\xi^{(k-1)} + \lambda_\tau^{(k-1)} = R_{k-2} \\ \hat{y}_\xi^{(k)} - \theta^{(k)} - \hat{x}_\xi^{(0)}\theta^{(k-1)} - \hat{x}_\xi^{(k-1)}\theta^{(0)} + \hat{y}_\tau^{(k-1)} = S_{k-2} \end{cases}$$

where R_ℓ, S_ℓ (and later $\tilde{R}_\ell$, $\tilde{S}_\ell$) depend only on $P^{(j)}$, $Q^{(j)}$ for $j \leq \ell$.

Thus $P_\xi^{(k)} + Q_{\xi\xi}^{(k)} = \tilde{R}_{k-1}$ and again, by subtraction,

$$(2.18) \qquad 2\lambda_\tau^{(k-1)} + 3(\lambda^{(0)}\lambda^{(k-1)})_\xi + 3\lambda_\xi^{(k-1)} - \frac{1}{3}\lambda_{\xi\xi\xi}^{(k-1)} = \dot{S}_{k-2}$$

where we used $P_\xi^{(k-1)} + Q_{\xi\xi}^{(k-1)} = \tilde{R}_{k-2}$ to eliminate $\hat{x}^{(k-1)}$ in (2.18).
Inductively, we find a formal solution using the power series (2.13) by
solving (2.18) for $\lambda^{(k-1)}$ and then obtaining $\hat{x}^{(k-1)}$ by the relation
$P_\xi^{(k-1)} + Q_{\xi\xi}^{(k-1)} = \tilde{R}_{k-2}$.

The nontrivial step is solving (2.18), the linearized KdV equation
with inhomogeneous terms. For water wave solutions of the system (2.3)
which behave like N-tuple solitary waves, we would choose for $\lambda^{(0)}$ an
N-soliton solution of the KdV equation and then solve (2.18) with this
$\lambda^{(0)}$, seeking solutions with the appropriate asymptotic behavior.

C.   Reformulation as a Mapping

We may formulate the equations in (2.3), somewhat artificially, as
components of a mapping $F(\varepsilon,\theta,\hat{y})$, where the equations (2.3) correspond
to $F(\varepsilon,\theta,\hat{y}) = 0$. This will be the analogue of Beale's approach in the
stationary case [4]. We begin with (2.3), ignoring the fact that this
is the rescaled version of (2.8).

Thus, we consider functions $\hat{y}(\xi,\eta,\tau)$ and $\theta(\xi,\eta,\tau)$ satisfying
$\hat{y}(\xi,0,\tau) = 0$, $\theta(\xi,0,\tau) = 0$ and define:

$$(2.19) \qquad \hat{x}(\xi,\eta,\tau) = \int_{-\infty}^{\xi} \hat{y}_\eta(\xi',\eta,\tau)\, d\xi' \quad ;$$

$$\lambda(\xi,\eta,\tau) = -\int_{-\infty}^{\xi} \theta_\eta(\xi',\eta,\tau)\, d\xi' \quad .$$

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The vector of functions $F(\varepsilon,\theta,\hat{y}) = (F_1,F_2,F_3,F_4)$ is defined as follows (note: F_1, F_2 are functions of ξ,η,τ while F_3, F_4 depend only on ξ,τ):

$$(2.20)\quad
\begin{cases}
F_1 \equiv \varepsilon\theta_{\xi\xi} + \theta_{\eta\eta} \;;\quad F_2 \equiv \varepsilon\hat{y}_{\xi\xi} + \hat{y}_{\eta\eta} \\[2ex]
F_3 \equiv \lambda_\xi e^{2\varepsilon\lambda} + \hat{y}_\xi e^{-3\varepsilon} + e^{\varepsilon\lambda}\Big\{\cos(\varepsilon^{3/2}\theta)\,[\varepsilon\lambda_\tau + \varepsilon^2(\lambda_\tau\hat{x}_\xi - \lambda_\xi\hat{x}_\tau) \\[1ex]
\qquad\qquad + \varepsilon^3(\theta_\tau\hat{y}_\xi - \theta_\xi\hat{y}_\tau)] \\[1ex]
\qquad\qquad + \dfrac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}}\,[-\varepsilon^3\theta_\tau + \varepsilon^4(\lambda_\tau\hat{y}_\xi - \lambda_\xi\hat{y}_\tau - \theta_\tau\hat{x}_\xi + \theta_\xi\hat{x}_\tau)]\Big\} \\[2ex]
F_4 \equiv \hat{y}_\xi\cos(\varepsilon^{3/2}\theta) - (1+\varepsilon\hat{x}_\xi)\dfrac{\sin(\varepsilon^{3/2}\theta)}{\varepsilon^{3/2}} + e^{-\varepsilon\lambda}\Big\{\varepsilon\hat{y}_\tau + \varepsilon^2(\hat{x}_\xi\hat{y}_\tau - \hat{x}_\tau\hat{y}_\xi)\Big\}
\end{cases}$$

where F_3, F_4 are evaluated at $\eta = 1$.

The degeneracy at $\varepsilon = 0$ which we observed in the formal expansion above ($\lambda^{(j)}$, $\hat{x}^{(j)}$ were only determined by order ε^{j+1} equations) may be removed by considering a modified operator $\tilde{F}(\varepsilon,\theta,\hat{y})$, which has the same roots as F for $\varepsilon > 0$.

We examine the kernel of $F(0,\theta,\hat{y})$: If $F(0,\theta,\hat{y}) = 0$, then $\theta_{\eta\eta} = 0$; $\hat{y}_{\eta\eta} = 0$; $\hat{y}_\xi - \theta\big|_{\eta=1} = 0$; $\hat{y}_\xi - \theta_\eta\big|_{\eta=1} = 0$. Recalling that $\hat{y}\big|_{\eta=0} = \theta\big|_{\eta=0} = 0$, this gives the solutions:

$$(2.21)\qquad \hat{y} = G(\xi,\tau)\cdot\eta \;;\quad \theta = G_\xi(\xi,\tau)\,\eta$$

where $G(\xi,\tau)$ is arbitrary except for the asymptotic condition: $G(\xi,\tau) \to 0$ as $|\xi| \to \infty$. To study the range of $F(0,\theta,\hat{y})$, suppose $F(0,\theta,\hat{y}) = (\alpha,\beta,r,s)$. Then since $\theta_\eta - \theta\big|_{\eta=1} = \int_0^1 \eta\cdot\theta_{\eta\eta}\,d\eta = \int_0^1 \eta\cdot\alpha\,d\eta$, we must have the compatibility condition:

$$(2.22)\qquad\qquad s - r = \int_0^1 \eta\cdot\alpha\,d\eta\;.$$

If we suppose that α,β,r are arbitrary functions which vanish as $|\xi| \to \infty$ for τ fixed, and define $s(\xi,\tau)$ by (2.22), i.e.

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$s = r + \int_0^1 \alpha \cdot \eta \; d\eta$ then $F(0,\tilde{\theta},\tilde{y}) = (\alpha,\beta,r,s)$ if

$$\tilde{\theta}(\xi,\eta,\tau) = \int_0^\eta (\eta-\eta') \; \alpha(\xi,\eta',\tau) \; d\eta' + A(\xi,\tau)\eta$$

(2.23)

$$\tilde{y}(\xi,\eta,\tau) = \int_0^\eta (\eta-\eta') \; \beta(\xi,\eta',\tau) \; d\eta' + B(\xi,\tau)\eta$$

where $B_\xi(\xi,\tau) - A(\xi,\tau) = s(\xi,\tau) - \int_0^1 (1-\eta')(\beta_\xi - \alpha) \; d\eta'$.

We will define a modified operator $\tilde{F}$ by means of a projection Q on the range space which incorporates the solvability condition (2.22). Define

$$(2.24) \quad Q(\alpha,\beta,r,s) = (0,0,0, \; s - r - \int_0^1 \alpha \cdot \eta \; d\eta)$$

Clearly Q is a projection and $QF(0,\theta,\hat{y}) \equiv 0$. Defining $\tilde{F}(\epsilon,\theta,\hat{y})$ as:

$$(2.25) \quad \tilde{F}(\epsilon,\theta,\hat{y}) = \begin{cases} \dfrac{QF(\epsilon,\theta,\hat{y})}{\epsilon} + (I-Q) \; F(\epsilon,\theta,y) \; , & \epsilon > 0 \\ \\ QF_\epsilon(0,\theta,\hat{y}) + (I-Q) \; F(0,\theta,y) \; , & \epsilon = 0 \end{cases}$$

$\tilde{F}$ is a smooth operator with the same roots as F for $\epsilon > 0$, but the additional condition $QF_\epsilon = 0$ leads to more regular behavior at $\epsilon = 0$. In fact, the added condition is precisely the KdV equation -- i.e. suppose $\tilde{F}(0,\theta,\hat{y}) = 0$. Then $QF_\epsilon(0,\theta,\hat{y}) = 0$ means precisely:

$$(2.26) \quad \hat{y}_\tau - \theta\hat{x}_\xi - \lambda_\tau + 3\hat{y}_\xi - 2\lambda\lambda_\xi - \int_0^1 \theta_{\xi\xi} \cdot \eta' \; d\eta' = 0 \quad \text{for} \quad \eta = 1.$$

Since $QF = 0$ at $\epsilon = 0$, $(I-Q)F(0,\theta,\hat{y}) = 0$ implies $\theta = G_\xi(\xi,\tau)$, $\hat{y} = G(\xi,\tau)\eta$. Using the relation $\lambda = - \int_{-\infty}^\xi \theta_\eta \; d\xi = -G(\xi,\tau)$ in (2.26) above, we obtain an equation for λ , namely:

$$(2.27) \quad - 2\lambda_\tau - 3\lambda\lambda_\xi - 3\lambda_\xi + \frac{1}{3}\lambda_{\xi\xi\xi} = 0$$

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where $\lambda = \lambda(\xi,\tau)$, $\hat{y} = -\lambda(\xi,\tau)\eta$, $\theta = -\lambda_\xi(\xi,\tau)\eta$. This is the KdV equation (2.16) above.

In the case of the solitary wave, Beale was able to find an appropriate scale of Banach spaces for the domain and range of $\tilde{F}$. He then proved that in a neighborhood of $\epsilon = 0$, $\theta = \theta_0$ (corresponding to the 1-soliton KdV solution), the Fréchet derivative of $\tilde{F}$ was invertible in the generalized sense of Nash-Moser. This implied the existence of a nontrivial solution for $\epsilon > 0$ sufficiently small by means of a generalized implicit function theorem (Zehnder [25]).

For the problem considered here, we have not as yet specified the proper function spaces. Continuing on a formal level, we consider the linearized operator at $\varepsilon = 0$, $\theta = \theta_0$, $\hat{y} = \hat{y}_0$, where $\tilde{F}(0,\theta_0,\hat{y}_0) = 0$.

We shall define another projection, P, mapping $(\theta,\hat{y})$ onto the kernel of $F(0,\cdot,\cdot)$ by:

(2.28) $P(\theta,\hat{y}) = (\hat{y}_\xi(\xi,1,\tau)\cdot\eta, \hat{y}(\xi,1,\tau)\cdot\eta)$

We note: $P(\theta,\hat{y}) = (0,0)$ if and only if $\hat{y}(\xi,1,\tau) \equiv 0$. Using the projections P and Q , we may regard the linearized operator $d\tilde{F}_{(0,\theta_0,\hat{y}_0)}$ as a matrix of operators:

(2.29) $$d\tilde{F} = \left(\begin{array}{cc} Q\ d\tilde{F}\big|_P & (I-Q)\ d\tilde{F}\big|_P \\[3mm] Q\ d\tilde{F}\big|_{I-P} & (I-Q)\ d\tilde{F}\big|_{I-P} \end{array}\right)$$

Since $(I-Q)\ d\tilde{F}\big|_P \equiv 0$, invertibility of $d\tilde{F}$ reduces to invertibility of the 'diagonal elements' $Q\ d\tilde{F}\big|_P$ and $(I-Q)\ d\tilde{F}\big|_{I-P}$. We remark that $(I-Q)\ \tilde{F}\big|_{I-P}$ is a linear operator, whose inverse we may compute explicitly. The equation $(I-Q)\tilde{F}\big|_{I-P}\ (0,\Theta,Y) = (\alpha,\beta,r,r+ \int_0^1 \eta\alpha\ d\eta)$ has the unique solution:

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(2.30) $\Theta(\xi,\eta,\tau) \equiv \int_0^\eta (\eta-\eta') \, \alpha(\xi,\eta',\tau) \, d\eta' - \eta[r(\xi,\tau) + \int_0^1 \alpha(\xi,\eta',\tau) \, d\eta']$

$Y(\xi,\eta,\tau) = \int_0^\eta (\eta-\eta') \, \beta(\xi,\eta',\tau) - \eta[\int_0^1 (1-\eta')\beta(\xi,\eta',\tau)d\eta']$

which is obtained by adding the condition $Y(\xi,1,\tau) \equiv 0$ to the solutions given previously in (2.23).

The essential difficulty with this inversion is that the Banach spaces for Θ, Y may include ξ and τ derivatives in their norms; for this reason, Beale uses a generalized implicit function theorem in the stationary case (shrinking the domains of analyticity in order to control derivatives).

The second diagonal element, $Q \, d\tilde{F}|_p$, is invertible if we can solve:

(2.31) $-2H_t - 3H_\xi - 3(\lambda \cdot H)_\xi + \frac{1}{3} H_{\xi\xi\xi} = J(\xi,\tau)$

the inhomogeneous linearized form of the KdV equation (see equation (2.18) above).

For the remainder of this paper, we consider the linearized KdV equation. As we have seen, it arises in the study of small amplitude, long wavelength, slow time variations of a steady flow of a perfect fluid over a flat bottom with Froude number near 1. If we seek solutions describing a 'nonlinear superposition' of N solitary waves of distinct speeds, the first approximant will be an N-soliton solution of the KdV equation and the higher order corrections will satisfy the inhomogeneous form of the linearized KdV equation (linearized about the N-soliton).

We shall consider the Cauchy problem for the linearized KdV equation. By Duhamel's principle, this amounts to solving the inhomogeneous equation. By the change of variables,

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(2.32)
$$X - 9T = \xi$$

$$- 6T = \tau$$

$$q(X,T) \equiv \frac{3}{2} \lambda(\xi,\tau)$$

we obtain the usual *form* of the KdV equation

(2.33)
$$q_T + q_{XXX} - 6qq_X = 0$$

We note that the τ-independent solution of the KdV equation is a function of $X - 9T$; this gives the one soliton with *speed* 9, which explicitly is $\lambda^{(0)} = -3\text{sech}^2(\frac{3}{2} \xi)$, the first order term on the expansion of Friedrichs and Hyers.

In the remaining sections, we shall use the letters x,y,t,u,v etc. for meanings other than those of the above section. Since these *different* meanings occur in separate places, this should cause no confusion for the reader.

3. SOME RESULTS ON THE CAUCHY PROBLEM FOR THE LINEARIZED KdV EQUATION

In this section, we summarize the results of [19] regarding the Cauchy problem:

$$(3.1) \qquad \begin{cases} u_t + u_{xxx} - 6(qu)_x = 0 \\ \\ u(x,0) = \phi(x) \end{cases}$$

where $q(x,t)$ satisfies the KdV equation

$$(3.2) \qquad q_t + q_{xxx} - 6qq_x = 0$$

By Duhamel's principle, the inhomogeneous form of (3.1) is solvable if the Cauchy problem is.

In [19], an explicit formula for the solution of problem (3.1) is given, using certain functions arising from the Schrödinger equation

$$(3.3) \qquad -f''(x,k,t) + q(x,t)f(x,k,t) = k^2 f(x,k,t)$$

where the potential $q(x,t)$ satisfies:

$$(3.4) \qquad \int_{-\infty}^{\infty} (1+x^2)|q(x,t)|dx < \infty \quad \text{for every } t \text{ fixed.}$$

The fundamental discovery of Gardner, Greene, Kruskal, and Miura [10], later formulated abstractly by Lax [15], is that if $q(x,t)$ evolves according to the KdV equation (3.2), the spectrum of the Schrödinger equation (3.3) is fixed and the associated scattering data evolves in a simple way. We shall use this information below, but first introduce some notation and basic facts about the scattering theory for (3.3). This information (and much more) may be found in [7].

Let $f_{\pm}(x,k,t)$ denote the Jost solutions of (3.3)

i.e. $f_+(x,k,t) \sim e^{ikx + 4ik^3 t}$ as $x \to +\infty$, t fixed

$\qquad f_-(x,k,t) \sim e^{-ikx - 4ik^3 t}$ as $x \to -\infty$, t fixed

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and both satisfy (3.3). We define the transmission coefficient, $T(k,t)$, in terms of the Wronskian of f_+, f_- as follows:

(3.5) $\dfrac{1}{T(k,t)} = \dfrac{1}{2ik} \, [f_+(x,k,t), \, f_-(x,k,t)]$

$= \dfrac{f'_+(x,k,t) \, f_-(x,k,t) - f'_-(x,k,t) \, f_+(x,k,t)}{2ik}$

(We shall always use the notation: $' \equiv \dfrac{\partial}{\partial x}$, $\cdot \equiv \dfrac{\partial}{\partial k}$.) It is not hard to show that $T(k,t) = T(k)$ is independent of t and that under the normalization of f_+, f_-, $T(k)$ is meromorphic in the upper half-plane Im $k > 0$ with poles at $k = i\beta_j$, $j = 1,\ldots,N$ where each energy $-\beta_j^2$ is a bound state energy for (3.3). N is finite by a classical estimate involving $\displaystyle\int_{-\infty}^{\infty} (1+|x|)\,|q(x)|\,dx < \infty$. $T(k)$ is also continuous and non-zero for real $k \neq 0$. For notational ease, we introduce for $j = 1,\ldots,N$ the following pair of functions:

(3.6) $F_j(x,t) = f_+^2(x,i\beta_j,t)$; $G_j(x,t) = c_j f_+(x,i\beta_j,t) \cdot g_j(x,t)$

where $g_j(x,t) = \dfrac{1}{i} \dfrac{d}{dk} \, [f_-(x,k,t) - \dfrac{f_-(x,i\beta_j,t)}{f_+(x,i\beta_j,t)} \, f_+(x,k,t)]\Big|_{k=i\beta_j}$ and c_j

is chosen so that $\displaystyle\int_{-\infty}^{\infty} F'_j(x,0) \, G_j(x,0) \, dx = 1$ for $j = 1,\ldots,N$.

The principal result of [19] is the following:

<u>Theorem 3.1</u> Suppose $q(x,t)$ satisfies (3.4). If $\phi(x)$ is continuous and integrable, the solution of (3.1) (in the sense of distributions) is given by:

(3.7) $u(x,t) = \displaystyle\int_{-\infty}^{\infty} \dfrac{dk}{4\pi ik} \; T^2(k) \left\{ \int_{-\infty}^{\infty} \dfrac{\partial}{\partial x} \, [f_+^2(x,k,t)\,f_-^2(y,k,0) \right.$

$\left. - f_-^2(x,k,t)\,f_+^2(y,k,0)]\,\phi(y)\;dy \right\}$

$\displaystyle\sum_{j=1}^{N} \int_{-\infty}^{\infty} [F'_j(x,t)\,G_j(y,0) - G'_j(x,t)\,F_j(y,0)]\,\phi(y)\;dy$

For a proof of this theorem, see [19].

Remark: It is known ([10], Theorem 3.6 or [19]) that the functions $(f_\pm^2)'(x,k,t)$, $F_j'(x,t)$, $G_j'(x,t)$ all satisfy the linearized KdV equation (3.1). The formula for $u(x,t)$ resembles the Fourier decomposition of $\phi(x)$, where the derivatives of the squared eigenfunctions replace the usual exponentials and the presence of a non-zero potential $q(x,t)$ can lead to the discrete terms $F_j'(x,t)$, $G_j'(x,t)$. In fact, when $q(x,t) \equiv 0$, (3.7) reduces to the usual Fourier transform solution of the Cauchy problem:

$$(3.8) \qquad \begin{cases} v_t + v_{xxx} = 0 \\ v(x,0) = \phi(x) \end{cases}$$

namely

$$(3.9) \qquad v(x,t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} dk \cdot e^{2ikx+8ik^3t} \left\{ \int_{-\infty}^{\infty} e^{-2iky} \phi(x) \, dy \right\}$$

since for $q \equiv 0$, $T(k) \equiv 1$ and $f_\pm(x,k,t) \equiv e^{\pm(ikx+4ik^3t)}$.

Noting that the solution $u(x,t)$ given by (3.7) consists of two pieces-- a discrete sum and an integral, we analyze them separately. The sum corresponds to variations in the soliton part of the function $q(x,t)$ and decomposes into travelling waves with positive velocities as t becomes large. For the water wave problem of Section 2, these terms are of considerable interest. The k-space integral part of (3.7) forms a dispersive wave train and will be seen to behave like the solution $v(x,t)$ of the Airy equation (3.8). In particular, for initial data which is somewhat smoother and more rapidly decaying than was assumed in Theorem 3.1 above, we show that this part of the solution $u(x,t)$ is smoother for $t > 0$ but, as $x \to -\infty$, it decays less rapidly. We present these results for linearizations about N-soliton solutions of the KdV equation. Similar analysis applies for a more general class of KdV solutions satisfying

(3.4); we omit such a discussion for the sake of brevity and restrict our attention to the N-soliton case.

Slower decay as $x \to -\infty$ for $t > 0$ occurs because of the dispersive nature of the oscillating solutions $\frac{d}{dx}(f_{\pm}^2(x,k,t))$ of the linearized KdV equation (see [24] for a general discussion of dispersive waves). In particular, the asymptotic behavior of $f_{\pm}^2(x,k,t)$, as $x \to \pm\infty$ respectively, is given by the exponentials $e^{\pm i\theta(x,k,t)}$, where we define:

$$(3.10) \qquad \theta(x,k,t) = 2kx + 8k^3 t.$$

These waves propogate with a negaLive velocity $-4k^2$ so that waves with large wave numbers contribute to the solution near $x = -\infty$ almost instantaneously.

The same expontentials, $e^{\pm i\theta}$, form the solution of the linearized equation for $q = 0$, (equation (3.9) above), namely

$$v_t + v_{xxx} = 0 ,$$

as is seen by Fourier transform, and arise in the asymptotics of $f_{\pm}^2(x,k,t)$, which, by the trace formula of Deift-Trubowitz [7], lead to a solution of the full KdV equation

$$(3.11) \qquad q_t + q_{xxx} - 6qq_x = 0.$$

(In [7], $q(x,0)$ is written as an integral over the real line in k:

$$(3.12) \qquad q(x,0) = \int_{-\infty}^{\infty} \frac{2i}{\pi} \, {\scriptstyle\sim}(k,0) f_+^2(x,k,0)\,dk + \sum_{j=1}^{N} a_j f_j^2(x,i\beta_j,0) .$$

An approach to the KdV equation itself using (3.12) will appear in a subsequent paper by the author.) The smoothness and decay properties of the solution of the Cauchy problem for the KdV equation were analyzed by Tanaka [23] and later Cohen Murray [5] using Faddeev-Marchenko inverse scattering theory rather than the then-unknown trace formula (3.12);

asymptotic analysis of the KdV equation also appeared in [1,2], where the more delicate regions $\frac{x}{t} = 0(1)$ as $t \to +\infty$ were also discussed in the absence of solitons.

Our analysis for the linearized KdV equation proceeds in direct analogy with the equation (3.8); the chief difference is the presence of the factors $m_{\pm}^2(x,k,t)$ multiplying the exponentials and their derivatives, which must be considered in all arguments. The techniques used will be primarily integration by parts and stationary phase analysis. In the limits we consider, the stationary phase points tend to $\pm\infty$, which complicates matters slightly. As in [5], we will work in a shrinking neighborhood of the stationary phase points, whose size is proportional to a small negative power of $|x|$. This variation of the usual stationary phase argument [8] is used to control the error terms arising at the stationary phase points. The smoothness argument relies on the observation [5] that for $t > 0$, we may rewrite the x-derivative of θ in terms of the k-derivatives of θ as follows:

$$(3.13) \qquad (\theta_x)^2 = 4k^2 = \frac{\theta_k}{6t} - \frac{x}{3t} .$$

We will use this to re-express u_{xx} as a function which is smoother than u_{xx} might otherwise appear to be.

Our results are summarized in the following theorem:

THEOREM 3.2   Assume $\phi(x)$, the initial data for the linearized KdV equation (3.1), has four continuous derivatives and that, for some fixed

$$(1 + |x|^\lambda)\left(\frac{d}{dx}\right)^\alpha \phi(x) \in L^1 \quad \text{for} \quad 0 \le \alpha \le 4.$$

when $q(x,t)$ is an N-soliton solution of the KdV equation, defined in (3.7) above, has the following properties:

$$(3.15) \begin{cases} \text{(i)} & u(x,t) \text{ is a classical solution of (3.1) for } t > 0 \\ & \text{with } u(x,0) = \phi(x) \\ \text{(ii)} & \partial_t^r \partial_x^s u(x,t) \text{ is continuous for } t > 0 \text{ for all non-negative} \\ & \text{integers } r,s \text{ satisfying } 3r + s \le 2\ell + 2 \\ \text{(iii)} & \lim_{x \to +\infty} |u(x,t) \cdot x^\ell| = 0 \qquad t > 0 \text{ fixed} \\ \text{(iv)} & |x|^{9/4} |u(x,t)| \text{ is bounded as a function of } x \text{ for } t > 0 \text{ fixed} \\ & \text{(even as } x \to -\infty) \\ \text{(v)} & |u(ct+\delta,t)| t^{1/2} \text{ is bounded for } c < 0 \text{ as } t \to +\infty, \quad \delta \text{ fixed.} \end{cases}$$

The proof of Theorem 3.2 is given in the three lemmas below, in which the smoothness and the limiting behavior are discussed separately. First, we present some facts concerning the Jost functions $f_\pm(x,k,t)$ in the N-soliton case, where we choose the phases of the waves so that $q(-x,-t) = q(x,t)$. (Recalling the scaling done in Section 2, this normalization is reasonable.)

The explicit form of the N-soliton leads to an algebraic expression for the Jost functions (see also [6]). In the proof of Theorem 3.2, we shall exploit certain properties of these functions, which we state here and prove in the Appendix. Define, for $j = 1,\ldots,N$,

$$(3.16) \quad \xi_j = x - 4\beta_j^2 t \;;\; \psi_j = \begin{cases} \cosh(\beta_j \, \xi_j), & j \text{ odd} \\ \\ \sinh(\beta_j \, \xi_j), & j \text{ even} \end{cases}$$

and consider the $N \times N$ Wronskian determinant (in x):

$$(3.17) \quad w(x,t) \equiv \det \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_N \\ \psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_N^{(1)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \psi_1^{(N-1)} & \psi_2^{(N-1)} & & \psi_N^{(N-1)} \end{pmatrix}$$

$$\equiv W_N(\psi_1, \ldots, \psi_N)$$

In the Appendix, we show $w(x,t) > 0$. The N-soliton solution of the KdV equation is given by:

$$(3.18) \quad q(x,t) \equiv -2 \frac{d^2}{dx^2} \log w(x,t)$$

while the eigenfunctions $f_\pm(x,k,t)$ are given by the ratios:

$$(3.19) \quad f_\pm(x,k,t) \equiv \frac{W_{N+1}\left[\psi_1, \psi_2, \ldots, \psi_N, e^{\pm i(kx+4k^3t)}\right]}{w(x,t) \prod_{j=1}^{N} \pm(ik - \beta_j)}$$

where W_{N+1} is the $(N+1) \times (N+1)$ Wronskian determinant. Writing $f_\pm(x,k,t) = m_\pm(x,k,t) e^{\pm i\theta(x,k,t)}$, we deduce the following properties of the factors $m_\pm(x,k,t)$ from (3.19):

(3.20)   (i) $m_\pm(x,k,t)$ are rational functions of k. Their

denominations and numerators are polynomials of degree

N in k ; both denominators are in fact precisely

$\prod_{j=1}^{N}(k + i\beta_j)$ while the numerators are polynomials

$k^N + A_1^\pm(x,t)k^{N-1} + \ldots + A_{N-1}^\pm(x,t)k + A_N^\pm(x,t)$ where

each coefficient $A_\ell^\pm(x,t)$ is a rational function of

$\left\{ e^{\beta_j \xi_j} \right\}$ which is bounded. The denominator of each

$A_\ell^\pm(x,t)$ is $w(x,t)$, which we show in the appendix is

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a sum of terms $e^{\sum\limits_{j=1}^{\ell} \epsilon_j \beta_j \xi_j}$ over all possible choices

$\epsilon_j = \pm 1$ with positive coefficients for each term.

(ii) $\dfrac{d}{dk} m_{\pm}(x,k,t)$ is a rational function of k which decays
like $|k|^{-2}$ as $|k| \to \infty$.

(iii) $\left(\dfrac{d}{dx}\right)^j m_{\pm}(x,k,t) = \dfrac{\sum\limits_{\ell=1}^{N} \left(\dfrac{d}{dx}\right)^j \left[A_{\ell}{\pm}(x,t)\right] \cdot k^{N-\ell}}{\prod\limits_{j=1}^{N}(k + i\beta_j)}$

so all x-derivatives of $m_{\pm}$ decay like $|k|^{-1}$ as
$|k| \to \infty$.

(iv) We also have: $T(k) \equiv \prod\limits_{j=1}^{N} \dfrac{k+i\beta_j}{k-i\beta_j}$

(3.21) $F_j{}'(x,t)$ and $G_j{}'(x,t)$ are real analytic in x,t and for fixed t,
they decay exponentially fast as $|x| \to \infty$ (see Appendix).

By (3.21), all the smoothness and decay properties of Theorem 3.2
are satisfied by $F_j{}'(x,t)$ and $G_j{}'(x,t)$. Therefore, we consider the
function $\tilde{u}(x,t)$ given by:

(3.22) $\tilde{u}(x,t) \equiv \displaystyle\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \dfrac{T^2(k)}{4\pi i k} \dfrac{d}{dx} \left\{ f_+^2(x,k,t)\, f_-^2(y,k,0) - f_-^2(x,k,t)\, f_+^2(y,k,0) \right\}$

$\cdot \phi(y)\,dy\,dk.$

Note that the integrand is continuous, even at $k = 0$ (since $f_+(x,0,t)$,
$f_-(x,0,t)$ are linearly dependent). Formula (3.22) suggests the following
definitions:

(3.23) $\tilde{\phi}_{\pm}(k) \equiv \displaystyle\int\limits_{-\infty}^{\infty} f_{\pm}^2(y,k,0)\,\phi(y)\,dy$

$= \displaystyle\int\limits_{-\infty}^{\infty} m_{\pm}^2(y,k,0)\, e^{\pm 2iky}\,\phi(y)\,dy$

We will analyze $\tilde{\phi}_\pm(k)$ just as in the usual Fourier transform case, using (3.20) to control the extra terms. Thus we shall see that $\tilde{u}(x,t)$, given by (3.22), and $v(x,t)$, the solution to the linearized problem for $q \equiv 0$ given by (3.9), behave quite similarly.

The first part of Theorem 3.2 is contained in the following lemma.

__Lemma 3.3__   If $(1 + |x|^\ell)\left(\dfrac{d}{dx}\right)^\alpha \phi(x) \in L^1$ for all $0 \le \alpha \le 4$ where $\ell \ge 4$ is fixed, then the functions $\partial_t^r \partial_x^s \tilde{u}(x,t)$, where $\tilde{u}(x,t)$ is given by (3.17) are continuous for all non-negative integers r,s satisfying $3r + s \le 2\ell + 2$.

__Proof of Lemma 3.3.__   The idea of the proof is a follows: we show that $\tilde{\phi}_\pm(k)$ decay rapidly enough as $|k| \to \infty$ that we can differentiate (4.9) twice with respect to x and still have a convergent integral. Then, using (3.13) to eliminate the $-4k^2$ factor arising from the exponentials and the estimate (3.20)(iii) to control derivatives of $m_\pm(x,k,t)$, we show that the integral for $\tilde{u}_{xx}$ can be differentiated twice. Repeating this argument, we obtain the desired result.

__Step 1.__   We show that $\left(\dfrac{d}{dk}\right)^\gamma \tilde{\phi}_\pm(k)$ exists and is $O(|k|^{-4})$ as $k \to \infty$ for $0 \le \gamma \le \ell$.

__Proof of Step 1.__   If we integrate (3.23) by parts, we have:

$$(3.24) \qquad \tilde{\phi}_\pm(k) = \frac{1}{\pm 2ik} \int_\infty^\infty \frac{d}{dy} (m_\pm^2(y,k,0)\phi(y))e^{\pm 2iky}dy.$$

The integral is absolutely convergent by our assumptions on ϕ and the properties of $m_\pm(y,k,0)$ listed above. In fact, we may integrate by parts four times with respect to y, obtaining

$$(3.25) \qquad \tilde{\phi}_\pm(k) = \frac{1}{(2ik)^4} \int_{-\infty}^\infty \left(\frac{d}{dy}\right)^4 (m_\pm^2(y,k,0)\phi(y))e^{\pm 2iky}dy$$

and the integral is still absolutely convergent. Thus $\tilde{\phi}_\pm(k)$ is $O(|k|^{-4})$ as $|k| \to \infty$.

Since $(1 + |x|^\ell)\phi(x) \in L^1$, the k-derivatives of $\tilde{\phi}_\pm(k)$ of order less

than or equal to ℓ all exist. Integrating these expressions by parts four times, we find

$$(3.26) \qquad \left(\frac{d}{dk}\right)^{\gamma} \tilde{\phi}_{\pm}(k) = O(|k|^{-4}) \quad \text{as} \quad |k| \to \infty \quad \text{for} \quad 0 \le \gamma \le \ell,$$

which completes Step 1.

<u>Step 2.</u> (The smoothness for $t > 0$). Writing

$$(3.27) \qquad \tilde{u}(x,t) \equiv \int_{-\infty}^{\infty} \frac{dk}{4\pi ik} T^2(k) \frac{d}{dx} [m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k)) - (m_-^2(x,k,t) e^{-i\theta} \tilde{\phi}_+(k))]$$

we have a continuous integrand which decays like $|k|^{-4}$ as $|k| \to \infty$. Therefore we may differentiate twice with respect to x and still have a convergent integral. This yields:

$$(3.28) \qquad \tilde{u}_{xx}(x,t) = \int_{-\infty}^{\infty} \frac{dk}{4\pi ik} T^2(k) \left(\frac{d}{dx}\right)^3 \left\{ m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k) - m_-^2(x,k,t) e^{-i\theta} \tilde{\phi}_+(k) \right\}$$

Since $\theta_x = 2k$ and $m_{\pm}'(x,k,t)$ decays like $|k|^{-1}$, the terms on the integrand in which $m_{\pm}'(x,k,t)$ or a higher derivative appears all have decay like $|k|^{-4}$ or faster; these terms can therefore be differentiated twice more with respect to x. The remaining terms, in which the exponential is differentiated three times, are:

$$(3.29) \qquad \int_{-\infty}^{\infty} \frac{T^2(k)}{2\pi} (-4k^2) \left\{ m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k) + m_-^2(x,k,t) e^{-i\theta} \tilde{\phi}_+(k) \ dk \right\}$$

$$= \int_{-\infty}^{\infty} \frac{T^2(k)}{2\pi} \left(\frac{x}{3t} - \frac{\theta_k}{6t} \right) \left\{ m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k) + m_-^2(x,k,t) e^{-i\theta} \tilde{\phi}_+(k) \right\} \ dk$$

$$\text{by (3.13)}$$

$$= \int_{-\infty}^{\infty} \frac{T^2(k)}{2\pi} \frac{x}{3t} \left\{ m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k) + m_-^2(x,k,t) e^{-i\theta} \tilde{\phi}_+(k) \right\} \ dk$$

$$+ \frac{1}{12\pi it} \int_{-\infty}^{\infty} \left\{ \left[\frac{d}{dk} (T^2(k) m_+^2(x,k,t) \tilde{\phi}_-(k)) \right] e^{i\theta} - \left[\frac{d}{dk} (T^2(k) m_-^2(x,k,t) \tilde{\phi}_+(k)) \right] e^{-i\theta} \right\} \ dk$$

where we integrated by parts in the second term. Each of these terms has continuous k-integrands which decay like $|k|^{-4}$ or better; hence they are also twice differentiable with respect to x. Therefore $\tilde{u}(x,t)$ has in fact four continuous x-derivatives for $t > 0$. Repeating this argument iteratively, we obtain $u(x,t)$ has $2\ell + 2$ continuous x-derivatives (since we can only bound $\left(\dfrac{d}{dk}\right)^{\gamma} \tilde{\phi}_{\pm}(k)$ for $0 \le \gamma \le \ell$, we may repeat the argument ℓ times).

To handle t-derivatives, we note that differentiating directly in t brings down a factor $\theta_t \equiv 8k^3$, which does not a priori lead to a convergent integral. However, multiplication by $8k^3$ may be expressed in the sense of distributions as $\left(\dfrac{\partial}{\partial x}\right)^3$ plus convergent integrals. Since $\left(\dfrac{\partial}{\partial x}\right)^3 \tilde{u}(x,t)$ is continuous, so is $\tilde{u}_t$. The equation (*) then gives higher regularity and the desired result. This proves Lemma 3.3.

The decay as $x \to +\infty$, $t > 0$ fixed and finite is given by:

Lemma 3.4. For $t > 0$ finite, fixed, $|u(x,t) x^{\ell}| \to 0$ as $x \to +\infty$.

Proof. Once again, we need only consider $\tilde{u}(x,t)$ since $F_j'(x,t)$, $G_j'(x,t)$ decay exponentially. For $x > 0$, $t > 0$ we note that $\theta_k = 2x + 24k^2 t > 0$ and in fact:

$$(3.30) \qquad \left| \frac{\theta_{kk}}{\theta_k} \right| = \left| \frac{48kt}{2x + 24k^2 t} \right| \le \left(\frac{x}{12t} \right)^{-1/2}.$$

Note also $\theta_{kkk} = 48t$, which is bounded. Write

$$(3.31) \qquad \tilde{u}(x,t) = \int_{-\infty}^{\infty} \rho(x,k,t) e^{i\theta} dk$$

by defining

$$(3.32) \qquad \rho(x,k,t)e^{i\theta} \equiv \frac{T^2(k)}{4\pi i k} \frac{d}{dx} \left[m_+^2(x,k,t) e^{i\theta} \tilde{\phi}_-(k) \right]$$

$$+ \frac{T^2(-k)}{4\pi i k} \frac{d}{dx} \left[m_-^2(x,-k,t) e^{i\theta} \tilde{\phi}_+(-k) \right].$$

We note $\rho(x,k,t)$ is continuous in k and decays like $|k|^{-4}$ as $|k| \to \infty$, as does $\left(\frac{\partial}{\partial k}\right)^\gamma \rho(x,k,t)$ for $0 \leq \gamma \leq \ell$.

Integrating (3.31) by parts ℓ times in k, we have

$$(3.33) \qquad \tilde{u}(x,t) = (i)^\ell \int_{-\infty}^{\infty} \left[\left(\frac{\partial}{\partial k} \frac{1}{\theta_k}\right)^\ell \rho(x,k,t) \right] e^{i\theta} \, dk.$$

Using (3.30) and the obvious bound $\left(\frac{1}{\theta_k}\right) \leq \frac{1}{2x}$ for $x > 0$ we obtain an estimate, for $t > 0$ fixed,

$$(3.34) \qquad |\tilde{u}(x,t)| \leq (2x)^{-\ell} C(t) \quad \text{for} \quad x \geq M > 0$$

where $C(t)$ is polynomial in $t^{1/2}$ of degree at most $\ell - 1$.

Moreover, since the integrand in (3.33) is integrable, by a simple modification of the usual Riemann-Lebesque lemma (namely, pick κ with $\int_{-\infty}^{-\kappa} + \int_{\kappa}^{\infty} < \varepsilon$ then approximate by a smooth function and integrate by parts), we can show $x^\ell \tilde{u}(x,t) \to 0$ as $x \to +\infty$ for $t > 0$ fixed, which proves lemma 3.4.

Finally we discuss decay as $x \to -\infty$ for $t > 0$ fixed:

Lemma 3.5. As $x \to -\infty$ for $t > 0$ fixed, $\tilde{u}(x,t) \cdot |x|^{9/4}$ remains bounded.

Proof of Lemma 3.5. Define $\alpha \equiv \left(\frac{-x}{12t}\right)^{1/2}$. For $x < 0$, $t > 0$, $\theta_k = 2x + 24k^2 t = 0$ for $k = \pm\alpha$. As $x \to -\infty$, (for fixed $t > 0$), $\alpha \to +\infty$. Let $k = \alpha\kappa$. Then

$$(3.35) \qquad \tilde{u}(x,t) = \int_{-\infty}^{\infty} \rho(x,k,t) e^{i\theta} dk = \alpha \int_{-\infty}^{\infty} \rho(x,\alpha\kappa,t) e^{i\lambda\tilde{\theta}} d\kappa$$

where $\lambda \equiv 2\frac{|x|^{3/2}}{(12t)^{1/2}}$ and $\tilde{\theta}(\kappa) \equiv -\kappa + \kappa^3/3$, so that $\tilde{\theta}(\kappa) = 0$ for $\kappa = \pm 1$.

In the usual stationary phase method, the chief contribution to the integral comes from the terms

$$\rho(x,\alpha,t) \int_1^{1+\varepsilon} e^{i\lambda[-\kappa+\kappa^3/3]} d\kappa \quad \text{and} \quad \int_1^{-1+\varepsilon} \rho(x,\alpha,t)e^{i\lambda\tilde\theta} d\kappa$$

where the κ value is frozen at $\kappa = \pm 1$ in the function ρ. The extra term $\int [\rho(x,\alpha\kappa,t) - \rho(x,\alpha,t)]e^{i\lambda\tilde\theta} d\kappa$ is of lower order for large λ by bounds on the derivative. In the case considered here, $\alpha \to \infty$ so this error term may become large. To counteract this, as in [5], we consider a very small interval about the stationary phase points $\kappa = \pm 1$, of order $|x|^{-\upsilon}$ for instance. We estimate as follows:

$$(3.36) \qquad \alpha \int_1^{1+\varepsilon} \rho(x,\alpha\kappa,t) e^{i\lambda\tilde\theta} d\kappa = \alpha\left[\int_1^{1+\varepsilon} \rho(x,\alpha,t)e^{i\lambda\tilde\theta} d\kappa \right.$$

$$\left. + \int_1^{1+\varepsilon} (\rho(x,\alpha\kappa,t) - \rho(x,\alpha,t))e^{i\lambda\tilde\theta} d\kappa \right]$$

$$= \alpha\rho(x,\alpha,t) \int_1^{1+\varepsilon} e^{i\lambda\tilde\theta} d\kappa$$

$$+ \alpha^2 \cdot \frac{d}{dk} \rho(x,\alpha\bar\kappa,t) \int_1^{1+\varepsilon} (\kappa-1)e^{i\lambda\tilde\theta} dk$$

for some $\bar\kappa \in [1, 1+\varepsilon]$

The first term is estimated as in the usual method of stationary phase; the second term leads after an integration by parts to an estimate of the form $C \cdot \alpha^{-2}\lambda^{-1}$, since $\frac{d}{dk}\rho$ decays like $|k|^{-4}$. With $\varepsilon = |x|^{-\upsilon}$, $0 < \upsilon < 1/2$, this term is of order $|x|^{-5/2-\upsilon}$ as $|x| \to \infty$. The first term decays like $\alpha^{-3} \cdot \lambda^{-1/2} = 0(|x|^{-9/4})$ as $|x| \to \infty$ and is the leading term.

Similar estimates hold on the intervals $[1-\varepsilon,1]$, $[-1-\varepsilon,-1]$, $[-1,-1+\varepsilon]$.

On the interval $[1+\varepsilon,\infty)$, we estimate as follows:

$$(3.37) \qquad \int_{1+\varepsilon}^{\infty} \rho(x,\alpha\kappa,t)e^{i\lambda\tilde{\theta}}d\kappa = \frac{-\alpha\, e^{i\lambda[-2/3+\varepsilon^2+\varepsilon^3/3]}}{i\lambda\varepsilon(2+\varepsilon)}\, \rho(x,\alpha(1+\varepsilon),t)$$

$$- \frac{\alpha}{i\lambda} \int_{1+\varepsilon}^{\infty} \left[\frac{d}{dk}\left(\frac{\rho(x,\alpha\kappa,t)}{\kappa^2-1}\right)\right]e^{i\lambda\tilde{\theta}}d\kappa.$$

which leads to a bound of the form:

$$(3.38) \qquad \left|\alpha \int_{1+\varepsilon}^{\infty} \rho(x,\alpha\kappa,t)e^{i\lambda\tilde{\theta}}d\kappa\right| \leq (\lambda^{-1}\alpha^{-3}\varepsilon^{-1})\cdot C_1$$

$$+ \lambda^{-1}\alpha^{-2}\varepsilon^{-1}\cdot C_2 + \lambda^{-1}\alpha^{-3}\varepsilon^{-2}\cdot C_3$$

$$= O(|x|^{-3+2\upsilon}) \quad \text{since} \quad 0 < \upsilon < 1/2$$

A similar estimate holds on the interval $(-\infty,-1-\varepsilon]$.

Finally, on the interval $[-1+\varepsilon,1-\varepsilon]$, we have:

$$(3.39) \qquad \alpha \int_{-1+\varepsilon}^{1-\varepsilon} \rho(x,\alpha\kappa,t)e^{i\lambda\tilde{\theta}}d\kappa = \frac{\alpha\cdot\rho(x,\alpha\kappa,t)e^{i\lambda\tilde{\theta}}}{i\lambda(\kappa^2-1)}\Bigg|_{\kappa=-1+\varepsilon}^{\kappa=1-\varepsilon}$$

$$- \frac{\alpha}{i\lambda} \int_{-1+\varepsilon}^{1-\varepsilon} \left[\frac{d}{dk}\left(\frac{\rho(x,\alpha\kappa,t)}{\kappa^2-1}\right)\right]e^{i\lambda\tilde{\theta}}d\kappa.$$

Integrating by parts three more times, we obtain:

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$$(3.40) \qquad \alpha \int_{-1+\varepsilon}^{1-\varepsilon} \rho(x,\alpha\kappa,t)e^{i\lambda\tilde{\theta}}d\kappa$$

$$= \alpha \left\{ \sum_{\ell=0}^{3} \left[\left(\frac{-1}{i\lambda(\kappa^2-1)} \frac{d}{d\kappa} \right)^{\ell} \frac{(x,\alpha\kappa,t)}{i\lambda(\kappa^2-1)} \right] \right\} e^{i\lambda\tilde{\theta}} \Bigg|_{\kappa=-1+\varepsilon}^{\kappa=1-\varepsilon}$$

$$+ \alpha \int_{-1+\varepsilon}^{1-\varepsilon} \left[\left(\frac{d}{d\kappa} \frac{1}{i\lambda(\kappa^2-1)} \right)^4 \rho(x,\alpha\kappa,t) \right] e^{i\lambda\tilde{\theta}} d\kappa.$$

The boundary terms lead to estimates, for $\ell = 0,1,2,3$ of the form:

$$\lambda^{-1}\alpha^{-3}\varepsilon^{-1} \ ; \ \lambda^{-2}\alpha^{-3}\varepsilon^{-3}(1+\alpha\varepsilon);$$

$$\lambda^{-3}\alpha^{-3}\varepsilon^{-5}(1+\alpha\varepsilon+\alpha^2\varepsilon^2) \ ; \ \lambda^{-4}\alpha^{-3}\varepsilon^{-7}(1+\alpha\varepsilon+\alpha^2\varepsilon^2+\alpha^3\varepsilon^3)$$

respectively.

Since α is $0(|x|^{1/2})$, λ is $0(|x|^{3/2})$ and $\varepsilon = |x|^{-\upsilon}$, the inequality $\upsilon < 1/2$ implies that the leading term is the first, i.e. $\lambda^{-1}\alpha^{-3}\varepsilon^{-1} = 0(|x|^{-3+\upsilon})$ as $|x| \to \infty$.

The *remaining integral*

$$\int_{-1+\varepsilon}^{1-\varepsilon} \left[\left(\frac{d}{d\kappa} \frac{1}{i\lambda(\kappa^2-1)} \right)^4 \rho(x,\alpha\kappa,t) \right] e^{i\lambda\tilde{\theta}} d\kappa$$

has a bound of the form:

$$C \lambda^{-4}(\alpha^5\varepsilon^{-4} + \alpha^4\varepsilon^{-5} + \alpha^3\varepsilon^{-6} + \alpha^2\varepsilon^{-7} + \alpha\varepsilon^{-8})$$

which has leading term

$$\lambda^{-4}\alpha^5\varepsilon^{-4} = 0\left(|x|^{-\frac{7}{2}+4\upsilon}\right) \quad \text{as} \quad |x| \to \infty.$$

This can be chosen to be of lower order than the contribution from the stationary phase points, by making

$$- \frac{7}{2} + 4\upsilon < - \frac{9}{4} \text{ , which is true for } \upsilon < \frac{5}{16}$$

This completes the proof of Lemma 3.5.

To finish the proof of Theorem 3.2, we remark that if $x = ct + \delta$, $c < 0$ then $\alpha \to \left(\frac{-c}{12} \right)^{1/2}$ as $t \to +\infty$, and the usual method of stationary phase applies [8]. This gives a decay rate of $\lambda^{-1/2}$ which is proportional to $t^{-1/2}$ and finishes the proof of Theorem 3.2.

4. GLOBAL BEHAVIOR AND UNIQUENESS FOR THE FIRST ORDER TERM IN THE N-SOLITON WATER WAVE PROBLEM

In section 2 above, a formal expansion procedure was given for the Euler equations for a fluid in a flat-bottomed canal which was near the constant horizontal flow of Froude number 1. We now show that the choice of an N-soliton solution of the KdV equation as leading term in this expansion results in an equation for the first order term which has a unique "N-tuple wave" solution if we add a symmetry requirement. As noted previously, this term satisfies the inhomogeneous form of the linearized KdV equation:

$$(4.1) \qquad Lu \equiv u_t + u_{xxx} - 6(qu)_x = h(x,t)$$

where $q(u,t)$ is an N-soliton and $h(x,t)$ is a term which depends only on $q(x,t)$. A simple calculation shows that in fact $h(x,t)$ is a linear combination of the functions:

$$q_x \, , \quad qq_x \, , \quad q_{xxx} \, , \quad q^2 q_x \, , \quad q\, q_{xxx} \, , \quad q_x \, q_{xx} \, , \quad q_{xxxxx} \, .$$

We remark that $h(x,t)$ contains terms of the form $F_j'(x,t)$, which satisfy the linearized equation. It is rather surprising that these "secular terms" [13] do not give rise to resonant solutions. The usual choice for the solution to $Lu = F_j'$ would be tF_j', which grows linearly in t in the moving frame in which $\xi_j \equiv x-4\beta_j^2 t$ remains constant as $t \to \infty$. However, the function $G_j'(x,t)$ is a solution of the homogeneous equation of the form (see the discussion in the Appendix)

$$(4.2) \qquad G_j'(x,t) = c_j [(x-12\beta_j^2 t)F_j'(x,t) + F_j(x,t) + H_j'(x,t)]$$

where $H_j'(x,t)$ is a rational function of the exponentials $\{e^{\beta_\ell \xi_\ell}\}$ which stays bounded for $t > 0$ fixed. Thus

(4.3) $\dfrac{G_j'(x,t)}{c_j} - (x - 4\beta_j^2 t)\, F_j'(x,t) - F_j(x,t) - H_j'(x,t) \equiv -8\beta_j^2 t\, F_j'(x,t)$

and since $L(G_j') = 0$, we have:

(4.4) $L[(x-4\beta_j^2 t)F_j'(x,t) + F_j(x,t) + H_j'(x,t)] = L[8\beta_j^2 t\, F_j'(x,t)]$

$$= 8\beta_j^2 F_j'(x,t)$$

The function $(x-4\beta_j^2 t)\, F_j'(x,t) + F_j(x,t) + H_j'(x,t)$ has the property that

it is bounded as $t \to \infty$ in any moving frame, even $\xi_j = $ constant, so we

have found a 'nonresonant' solution for the secular term $F_j'(x,t)$. For

the secular forcing terms $G_j'(x,t)$, the growth in the obvious solution is

quadratic in t as $t \to \infty$ with ξ_j fixed and, to the best of our

knowledge, no nonresonant solutions of (4.1) exist. By the absence of

these secular terms, the perturbation we consider is rather special.

In order to study (4.1) when $h(x,t)$ is a linear combination of the

functions listed above, we use the *following representation of the N-soliton*

solutions of the KdV equation (see Gardner, Greene, Kurskal, and Miura [10],

Tanaka [22], and Deift-Trubowitz [7]):

(4.5) $q(x,t) = \displaystyle\sum_{j=1}^{N} a_j\, F_j(x,t)$

where $F_j(x,t)$ is as usual the squared eigenfunction $f_+^2(x, i\beta_j, t)$. Then,

using the third order equation satisfied by the squared eigenfunctions [19],

we have:

$$(4.6) \qquad q''' = \sum_{j=1}^{N} a_j [4(q(x,t) + \beta_j^2) F_j' + 2q' F_j]$$

$$= 6qq' + \sum_{j=1}^{N} 4a_j \beta_j^2 F_j'$$

$$\therefore qq''' = 6q^2 q' + \sum_{j=1}^{N} 4a_j \beta_j^2 q \, F_j'$$

$$q^{(5)} = 6qq''' + 18 \, q'q'' + \sum_{j=1}^{N} [16a_j \beta_j^2 (q+\beta_j^2) F_j' + 8a_j \beta_j^2 q' F_j]$$

Thus our particular forcing term $h(x,t)$ is in the span of the functions:

$$(4.7) \qquad \left\{ F_j'(x,t), \; q(x,t)F_j'(x,t), \; q'(x,t)F_j(x,t), \; q^2 q'(x,t), \; q' \, q'' \right\}, \quad j=1,\ldots,N.$$

We prove:

Lemma 4.1. Suppose $h(x,t)$ is a linear combination of the functions in

(4.7). Then there exists a solution to the linearized KdV equation (4.1),

$Lu = h$, which is an N-tuple wave in the following sense:

(i) $u(x,t) \to 0$ exponentially fast as $|x| \to \infty$ for t fixed

(ii) $u(ct+\delta,t) \to 0$ exponentially fast as $t \to +\infty$

   if $c \neq 4\beta_j^2$, $j = 1,\ldots,N$

(iii) $\lim_{t \to \infty} u \, (4\beta_j^2 t+\xi,t)$ exists and is an exponentially decreasing

   function of ξ

(iv)   In fact, $u(x,t)$ is a sum of terms which are either

   (a) rational functions in the N-exponentials $\left\{ \exp \{ \beta_j (x - 4\beta_j^2 t) \} \right\}$

      with the same denominator $[w(x,t)]^2$ as $q(x,t)$ or

   (b) of the form $(x-4\beta_j^2 t) \, F_j'(x,t)$.

Proof of Lemma 4.1: As above, we write

$$(4.8) \qquad\qquad L(u) \equiv u_t + u_{xxx} - 6(qu)_x .$$

From [10], Theorem 3.6, $F'_j(x,t)$ satisfy $L(u) = 0$ and $F_j(x,t)$ satisfy the adjoint equation

$$(4.9) \qquad\qquad v_t + v_{xxx} - 6q\,v_x = 0$$

Thus $L(F_j) = -6F_j\,q'$; $L(tF'_j) = F'_j$. Also, $L\big((x-12\beta_j^2 t)F'_j + F_j\big) = 6q\,F'_j$. From the KdV equation, $L(q^2) = 6q'q'' - 6q^2 q'$ and $L(q'') = 12q'q''$. A basis for the solution is therefore given by

$$(4.10) \qquad \{F_j,\ tF'_j,\ (x-12\beta_j^2 t)F'_j + F_j,\ q^2,\ q''\}$$

Using $L(G'_j) = 0$ and (4.2), an equivalent basis is:

$$(4.11) \qquad \left\{F_j,\ (x-4\beta_j^2 t)F'_j,\ H'_j,\ q^2,\ q''\right\}$$

The functions F_j, q^2, q'' have properties (i)-(iv) of the lemma since $F_j(x,t)$ is exponentially decreasing. In the Appendix, we show that $(x-4\beta_j^2 t)\,F'_j(x,t)$ is bounded and satisfies (i)-(iv) and prove that $H'_j(x,t)$ is a rational function of the expontentials which has *properties* (i)-(iv) as well. Assuming these results, the lemma is proved.

<u>Remarks</u>: (i) Uniqueness: If we choose the phases of the N-soliton so that $q(-x,-t) = q(x,t)$ then the solution in Lemma 4 is *unique* provided we require:

(a) $\quad u(-x,-t) = u(x,t)$

(b) $\quad u(ct+\delta, t)$ is bounded for all c as $t \to \infty$

(c) $\quad u(ct+\delta, t) \to 0$ exponentially fast if $c \neq 4\beta_j^2$.

<u>Proof</u>: From the results of Section 3, the kernel of L is spanned by $F'_j(x,t)$, $G'_j(x,t)$ and $\{(f_\pm^2)'(x,k,t)\,|\,-\infty < k < \infty\}$. The functions F'_j violate (a); $G'_j(x,t)$ violates (b) for $c = 4\beta_j^2$; by stationary phase analysis, $(f_\pm^2)'(x,k,t)$ violates (c) for $c = -4k^2$ (the decay is algebraic, not exponential). Thus $u(x,t)$ as given in Lemma 4.1 is unique, since in this case, the functions given in (4.11) satisfy all these conditions.

(ii) Higher order terms: Even in the time-independent case, explicit expressions for the higher order terms of the formal expansion involve more complicated, transcendental functions. For even the second-order term, functions like

$$\log (1 - \tanh \beta x) \qquad \text{sech}^2 \beta x$$

occur. Thus algebraic methods will not readily yield solvability results like Lemma 4.1 for the higher order terms.

APPENDIX

In this appendix, we collect certain facts about N-solitons of the KdV equation and the associated eigenfunctions $f_+(x,k,t)$ in this case. These properties are well known ([6], [10], [13], [20], [22]), so we sketch the proofs for the most part. The functions $G_j(x,t)$ do not appear in these papers, so results regarding these eigenfunctions are presented and proved in full.

With the choice of phases so that $q(x,t) = q(-x,-t)$, the N-soliton $q(x,t)$ with bound states $-\beta_N^2 < -\beta_{N-1}^2 < \ldots < -\beta_1^2 < 0$ where $\beta_j > 0$, it is given explicitly by the following formulae:

Let $\xi_j \equiv x - 4\beta_j^2 t$ and define

(A.1)
$$\xi_j(x,t) = \begin{cases} \cosh(\beta_j \xi_j) & \text{if } j \text{ is odd} \\ \sinh(\beta_j \xi_j) & \text{if } j \text{ is even} \end{cases}$$

Let $w(x,t) \equiv W_N(\psi_1, \psi_2, \ldots, \psi_N)$

$$= \det \begin{pmatrix} \psi_1 & \psi_2 & \ldots & \psi_N \\ \psi_1^{(1)} & \psi_2^{(1)} & \ldots & \psi_N^{(1)} \\ \vdots & \vdots & & \\ \psi_1^{(N-1)} & \psi_2^{(N-1)} & & \psi_N^{(N-1)} \end{pmatrix},$$

the Wronskian determinant in x of $\psi_1, \ldots, \psi_N$.

Then

$$q(x,t) \equiv -2 \frac{d^2}{dx^2} \log w(x,t) .$$

This definition is sensible because $w(x,t) > 0$, which we show below.

<u>Lemma A.1.</u> $w(x,t) > 0$.   In fact, $w(x,t)$ is a sum of exponentials with positive coefficients.

<u>Proof</u>: $w(x,t) = W_N\Big(\cosh \beta_1\xi_1\, ,\, \sinh \beta_2\xi_2\, ,\, \ldots,$

$$\frac{\exp\{\beta_N\xi_N\} + (-1)^{N-1}\exp\{-\beta_N\xi_N\}}{2}\Big)$$

By the multilinearity of the determinant, this is the sum over all possible choices $\epsilon_j = \pm 1$ of $2^{-N} W_N\Big(\exp\{\epsilon_1\beta_1\xi_1\},\ \epsilon_2 \exp\{\epsilon_2\beta_2\xi_2\},\ \exp\{\epsilon_3\beta_3\xi_3\},$ $\epsilon_4 \exp\{\epsilon_4\beta_4\xi_4\}\ \ldots\Big)$ i.e.   we have upon evaluating the Vandermonde determinants,

(A.2) $w(x,t) = 2^{-N} \sum_{\substack{\text{all choices} \\ \epsilon_j = \pm 1}} \exp\{\big(\sum_{\ell=1}^{N} \epsilon_\ell\beta_\ell\epsilon_\ell\big)\}\ \epsilon_2\epsilon_4 \cdots \epsilon_{2[n/2]}$

$\cdot \prod_{j<k} (\epsilon_k\beta_k - \epsilon_j\beta_j)\ .$

Since $0 < \beta_1 < \beta_2 < \cdots < \beta_N$, the number of negative factors in the product can be found explicitly. Namely, if $\epsilon_{i_1},\ldots,\epsilon_{i_r}$ are the negative indices for a given choice of the ϵ's, we obtain $(i_1-1) + \cdots + (i_r-1)$ negative factors in $\prod_{j\,k} (\epsilon_k\beta_k - \epsilon_j\beta_j)$. The extra factor $\epsilon_2\epsilon_4\cdots\epsilon_{2[n/2]}$ adds an additional (-1) factor for each i_j which is even. For any choice of r and $i_1,\ldots,i_r$, this means that there are an even number of (-1) factors; thus every term in the sum (A.2) has a positive coefficient. We remark that all exponents $\sum_{j=1}^{N} \epsilon_j\beta_j\xi_j$ occur in $w(x,t)$ and that $w(x,t) = w(-x,-t)$ since changing $\{\epsilon_j\} \to \{-\epsilon_j\}$ does not alter the coefficients in (A.2). This proves Lemma A.1.

The eigenfunctions $f_+(x,k,t)$ are given explicitly by:

(A.3) $f_+(x,k,t) \equiv \dfrac{W_{N+1}(\psi_1,\psi_2,\ldots,\psi_N,\ \exp\{ikx+4ik^3t\})}{w(x,t) \prod_{\ell=1}^{N} (ik - \beta_\ell)}$

The normalization $f_+ \sim \exp\{ikx+4ik^3t\}$ as $x \to +\infty$ for t fixed is satisfied, as is seen by looking at the leading term, which has exponent $\sum_{j=1}^{N} \beta_j \xi_j$ (i.e. pick the term with all ε_k's $= +1$ in (A.2) and the corresponding expansion of the numerator). From the fact ([6], [7]) that $T(k) \equiv \prod_{j=1}^{N} (k+i\beta_j)/(k-i\beta_j)$, we obtain a similar expression for $f_-(x,k,t)$ using $T(k) f_-(x,k,t) \equiv f_+(x,k,t)$. The proof that $f_\pm(x,k,t)$ satisfy the Schrödinger equation with potential $q(x,t)$ defined as above is given in Deift [6]; the basic idea is to use Jacobi's identity for the Wronskians and induction on N.

From (A.3) and the expression for $f_-(x,k,t)$, it is easy to see that

$$f_+(x,i\beta_j,t) + (-1)^{j+1} f_-(x,i\beta_j,t) \equiv 0 .$$

Also, from (A.2) and (A.3), it is clear that $f_+(x,i\beta_j,t)$ decays like $\exp\{-\beta_j|\xi_j|\}$ as $|\xi_j| \to \infty$ since the exponentials $\exp\{\pm\beta_j\xi_j\}$ in the numerator will cancel each other, while remaining in the denominator $w(x,t)$.

The factor is $g_j(x,t)$ defined in (3.6) as

$$(A.4) \qquad g_j(x,t) \equiv \frac{1}{i} \frac{d}{dk} \left(f_-(x,k,t) + (-1)^{j+1} f_+(x,k,t) \right) \Big|_{k=i\beta_j}$$

Differentiating the exponential $\exp\{ikx+4ik^3t\}$ gives a term $2(x-12\beta_j^2 t) f_+(x,i\beta_j,t)$, while differentiating the factors $(\pm ik)^{\ell-1}$ which occur in the $(\ell,N+1)$ entry in the Wronskian gives terms having the form $\left(c_\varepsilon \exp\{\lambda(x,t)\} \right)/\left(w(x,t) \right)$ where $\lambda(x,t) \equiv \pm\beta_j\xi_j + \sum_{\ell=1}^{N} \varepsilon_\ell \beta_\ell \xi_\ell$ whose sum we denote by $h_j(x,t)$. Thus $g_j(x,t)$ grows like $\exp\{\beta_j|\xi_j|\}$ as $|\xi_j| \to \infty$ and is of the form $2(x-12\beta_j^2 t) f_+(x,i\beta_j,t) + h_j(x,t)$ where $h_j(x,t)$ is a rational function of the N exponentials $\exp\{\beta_\ell\xi_\ell\}$ with denominator $w(x,t)$, growing like $\exp\{\beta_j|\xi_j|\}$ as $|\xi_j| \to \infty$. Since $g_j(x,t) \equiv c_j f_+(x,i\beta_j,t) g_j(x,t)$, multiplying by $f_+(x,i\beta_j,t)$ we have

$$(A.5) \quad \frac{G_j(x,t)}{c_j} \equiv 2(x-12\beta_j^2 t) f_+^2(x,i\beta_j,t) + f_+(x,i\beta_j,t) h_j(x,t)$$

$$\equiv 2(x-12\beta_j^2 t) F_j(x,t) + H_j(x,t) \ .$$

Since $f_+(x,i\beta_j,t)$ is rational with denominator $w(x,t)$ and decays like $\exp\{-\beta_j|\xi_j|\}$ as $|\xi_j| \to \infty$, $H_j(x,t)$ is rational with denominator $(w(x,t))^2$ and is bounded as $|\xi_j| \to \infty$. Since all the other exponentials occur in the numerator with growth at most $\exp\{\sum_{\ell \neq j} 2\beta_\ell |\xi_\ell|\}$ and these terms are balanced by those in the denominator, $H_j(x,t)$ is bounded for all x,t real. It then follows that $H_j'(x,t)$ is a sum of terms which decrease exponentially fast as $t \to \infty$ except in the frames $x-4\beta_j^2 t$ = constant, where their limit is an exponentially decreasing function of the variable $\xi_j \equiv x-4\beta_j^2 t$. Note, however, that for $t \to \infty$, $H_j'(x,t)$ "decouples" into N exponentially decreasing bumps moving at the speeds $4\beta_j^2$ with the same phases as $q(x,t)$ as $t \to \infty$; unlike $F_j'(x,t)$, these terms give rise to contributions in all N moving frames. These are the basic properties used in the discussion in Sections 3 and 4 above.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TSR # 2208	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Justification of the KdV Approximation to First Order in the Case of N-Soliton Water Waves in a Canal		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert L. Sachs		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-77-01986
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1(Applied Analysis) 2(Physical Mathematics)
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE April 1981
		13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Euler equations; Korteweg-de Vries equation; N-Solitons

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the Euler equations for a perfect fluid in a flat-bottomed canal in the time-dependent case. A formal expansion procedure for small amplitude, long waves analogous to that of Friedrichs and Hyers for solitary waves is developed and leads to the Korteweg-de Vries equation (KdV for short) for the lowest order term. The higher order terms in the expansion satisfy the inhomogeneous version of the linearized KdV equation.

Of particular interest to us are those solutions of the KdV equation called N-solitons, which asymptotically separate into N travelling waves with

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20. Abstract, continued

distinct speeds. Using certain facts about the linearized KdV equation and some properties of the N-solitons, we prove that the next term in this expansion can be uniquely specified by certain asymptotic conditions and a symmetry requirement. This solution behaves like an N-soliton; asymptotically, it separates into N travelling waves with the same speeds and phases as those of the leading term.

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